The sixth Painlevé equation as similarity reduction of $\widehat{\mathfrak{gl}}_3$ hierarchy

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Abstract

Scaling symmetry of $\widehat{\mathfrak{gl}}_n$-type Drinfel'd-Sokolov hierarchy is investigated. Applying similarity reduction to the hierarchy, one can obtain the Schlesinger equation with $(n+1)$ regular singularities. Especially in the case of $n = 3$, the hierarchy contains the three-wave resonant system and the similarity reduction gives the generic case of the Painlevé VI equation. We also discuss Weyl group symmetry of the hierarchy.

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1 Introduction

There are close connections between soliton equations and the Painlevé equations. For instance, self-similar solutions of the modified KdV equation satisfy the second Painlevé equation, and that of the modified Boussinesq equation satisfy the fourth Painlevé equation. In these examples, soliton equations can be regarded as special cases of the $A_l^{(1)}$-type Drinfel'd-Sokolov hierarchies. Along this line, Noumi and Yamada obtained the $A_l^{(1)}$ Painlevé system [NY, N] as a similarity reduction of the Drinfel'd-Sokolov hierarchy of the principal $A_l^{(1)}$-type.

In recent works [KK1, KK2], we developed a technique that connects Lax formalism of soliton equations and monodromy preserving deformation, based on similarity reduction of the generalized Drinfel'd-Sokolov hierarchies. As consequences, the Painlevé IV equation can be obtained from derivative nonlinear Schrödinger equation [KK1, KK2], and the Painlevé V equation from the modified Yajima-Oikawa equation [KIK].

In this paper, extending these works, we will show a connection between the three-wave resonant system,

$$\begin{cases} \partial_\tau u_1 + c_1 \partial_x u_1 = i\gamma_1 u_2^* u_3^*, \\ \partial_\tau u_2 + c_2 \partial_x u_2 = i\gamma_2 u_3^* u_1^*, \\ \partial_\tau u_3 + c_3 \partial_x u_3 = i\gamma_3 u_1^* u_2^*, \end{cases} \tag{1.1}$$

and the Painlevé VI equation,

$$\begin{aligned} \frac{d^2y}{dt^2} =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}, \end{aligned} \tag{1.2}$$

where α, β, γ, δ are complex parameters.

The relation between the three-wave resonant system (1.1) and the Painlevé VI equation (1.2) has been studied by considering self-similar solutions of (1.1) [FY, K]. Fokas and Yortsos [FY] obtained one-parameter family of Painlevé VI transcendents. Based on this work, Kitaev obtained a two-parameter family [K]. In this paper, we introduce a hierarchy of soliton equations that includes the three-wave resonant system and show that the generic case of the Painlevé VI equation with four parameters can be obtained as similarity reduction of the hierarchy.

Our method to obtain the relation between (1.1) and (1.2) is based on a transformation that maps the Lax equations of the three-wave resonant system, whose coefficients are 3×3 matrices, into the Schlesinger system of the form,

$$\begin{cases} \dfrac{\partial Y(x)}{\partial x} = \mathcal{A}(x)Y(x), \quad \mathcal{A}(x) \stackrel{\text{def}}{=} \dfrac{\mathcal{A}_0}{x} + \dfrac{\mathcal{A}_1}{x-1} + \dfrac{\mathcal{A}_t}{x-t}, \\ \dfrac{\partial Y(x)}{\partial t} = -\dfrac{\mathcal{A}_t}{x-t}Y(x), \end{cases} \tag{1.3}$$

where A_i $(i = 0, 1, t)$ are 2×2 matrices independent of x and obey the following

conditions:

$$\begin{aligned} &\det \mathcal{A}_i = 0, \qquad \text{eigenvalues of } \mathcal{A}_i \text{ are } 0 \text{ and } \theta_i \quad (i = 0, 1, t), \\ &-\mathcal{A}_0 - \mathcal{A}_1 - \mathcal{A}_t = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \end{aligned} \tag{1.4}$$

It is well-known that the Painlevé VI equation is obtained from the Schlesinger system (1.3) [JM] (cf. [O1]). The zero of the $(1,2)$-element of the coefficient matrix $\mathcal{A}(x)$, i.e.,

$$y(t) = \frac{-t(\mathcal{A}_1)_{12}}{(\mathcal{A}_2)_{12} + t(\mathcal{A}_3)_{12}}, \tag{1.5}$$

solves the Painlevé VI equation (1.2) with the parameters,

$$\alpha = \frac{(\kappa_1 - \kappa_2 - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1 - \theta_t^2}{2}. \tag{1.6}$$

Harnad [H] and Mazzocco [M] gave the description of the Painlevé VI in terms of 3×3 linear system with a simple pole at 0 and a double pole at ∞;

$$z\frac{d\Psi}{dz} = (T + V)\Psi, \qquad T = \mathrm{diag}(0, t, 1) \tag{1.7}$$

They discussed the transformation of the 3×3 system (1.7) into the 2×2 system (1.3). In this paper, we will show that the 3×3 linear problem (1.7) and its Laplace transformation to (1.3) can be reformulated as similarity reduction of the Drinfel'd-Sokolov hierarchy of $\widehat{\mathfrak{gl}}_3$-type.

Note that if the matrix V in (1.7) is skew symmetric, the soliton equation associated with (1.7) is equivalent to the Darboux-Egoroff equation, which is a special case of (1.1) and characterize the metric behind the solution of the Witten-Dijkgraaf-Verlinde-Verlinde equations [Du, AvdL]. As discussed in [Du, AvdL], the deformation equations of (1.7) with skew symmetric V gives a one-parameter family ($\alpha = -\beta, \gamma = 0, \delta = 1/2$) of the Painlevé VI transcendents, whereas we do not impose the skew-symmetric condition for V in this paper. Therefore we obtain the generic case of the Painlevé VI.

This paper is organized as follows. In section 2, we review our construction of the soliton equations by using the affine Lie algebra $\widehat{\mathfrak{gl}}_n$. The three-wave resonant system (1.1) can be treated as the case $n = 3$. In section 3, we discuss scaling symmetry of the $\widehat{\mathfrak{gl}}_n$ hierarchy, and consider similarity reduction of the hierarchy. If we consider the $\widehat{\mathfrak{gl}}_3$ case, we can obtain the Painlevé VI equation with full parameters. Weyl group symmetry of the hierarchy is discussed in Section 4. Section 5 is devoted to concluding remarks.

2 $\widehat{\mathfrak{gl}}_n$ hierarchy and the three-wave resonant system

In this section we outline our formulation of soliton equations [KK1], which is based on the affine Lie group approach of Drinfel'd and Sokolov [DS] and its generalizations

[BtK2, dGHM, W]. We remark that we use only the homogeneous case in this paper, to treat the three-wave resonant system and the Painlevé VI equation.

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. The affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{gl}}_n$ is realized as a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ together with the derivation $d = d/dz$:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d. \tag{2.1}$$

The Lie bracket is given as follows:

$$\begin{aligned} &[X \otimes z^j, Y \otimes z^k] = [X, Y] \otimes z^{j+k} + j\delta_{j+k,0}(X, Y)K, \\ &[K, \widehat{\mathfrak{g}}] = 0, \qquad [d, X \otimes z^j] = jX \otimes z^j, \end{aligned} \tag{2.2}$$

for $X, Y \in \mathfrak{gl}_n$, $j, k \in \mathbb{Z}$ and (X, Y) is the normalized invariant scalar product of $\mathfrak{g}$ [K]. The derivation d induces a $\mathbb{Z}$-grading on $\widehat{\mathfrak{g}}$, which is called the homogeneous gradation:

$$\widehat{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \widehat{\mathfrak{g}}_j, \quad \widehat{\mathfrak{g}}_j = \{x \in \widehat{\mathfrak{g}} \mid [d, x] = jx\}. \tag{2.3}$$

For an integer k, we use the notation

$$\widehat{\mathfrak{g}}_{\geq k} = \bigoplus_{j \geq k} \widehat{\mathfrak{g}}_j, \quad \widehat{\mathfrak{g}}_{<k} = \bigoplus_{j<k} \widehat{\mathfrak{g}}_j. \tag{2.4}$$

To construct integrable hierarchies, Heisenberg subalgebras of $\widehat{\mathfrak{gl}}_n$ play a crucial role. In general, nonequivalent Heisenberg subalgebras are classified by conjugacy classes of the Weyl group of $\mathfrak{g}$ [dGHM]. In the case of $A_l^{(1)}$, conjugacy classes are labeled by partitions $\{(n_1, \ldots, n_s); n_1 + \cdots + n_s = n\}$. In this paper, we consider the homogeneous Heisenberg subalgebra $\widehat{\mathfrak{s}}$ that is associated with the partition $(1, 1, \ldots, 1)$. In this case, we can choose a graded basis of $\widehat{\mathfrak{s}}$ as

$$\Lambda_k^a \stackrel{\text{def}}{=} z^k E_{aa} \qquad (1 \leq a \leq n,\ k \in \mathbb{Z} \setminus \{0\}) \tag{2.5}$$

where $E_{aa} = (\delta_{ia}\delta_{ja})$ is the matrix unit. These basis of $\widehat{\mathfrak{s}}$ are related to the homogeneous gradation:

$$[d, \Lambda_k^a] = k\Lambda_k^a. \tag{2.6}$$

We now consider a Kac-Moody group $\widehat{G}$ formed by exponentiating the action of $\widehat{\mathfrak{g}}$ on a integrable module. Throughout this paper, we assume that the exponentiated action of an element of the positive degree subalgebra of $\widehat{\mathfrak{s}}$ is well-defined. We remark that all of the representations used in what follows belong to this category. We denote by $\widehat{G}_{\geq 0}$ and $\widehat{G}_{<0}$ the subgroups of $\widehat{G}$ correspond to the subalgebras $\widehat{\mathfrak{g}}_{\geq 0}$ and $\widehat{\mathfrak{g}}_{<0}$, respectively.

We construct a hierarchy of soliton equations with time valuables $\boldsymbol{t} = (t_j^1, \ldots, t_j^n)_{j>0}$. First we define

$$\Psi_0(z; \boldsymbol{t}) \stackrel{\text{def}}{=} \exp\Big(\sum_{j>0} \sum_{a=1}^{n} t_j^a \Lambda_j^a\Big) = \exp\Big(\sum_{a=1}^{n} \Big(\sum_{j>0} z^j t_j^a\Big) E_{aa}\Big), \tag{2.7}$$

Starting from an element $g(z;0) \in \widehat{G}$, we define the time-evolution by

$$g(z;\boldsymbol{t}) \stackrel{\text{def}}{=} \Psi_0(z;\boldsymbol{t})g(z;0). \tag{2.8}$$

Then $g(z;\boldsymbol{t})$ satisfies the following differential equations,

$$\frac{\partial g(z;\boldsymbol{t})}{\partial t_j^a} = \Lambda_j^a g(z;\boldsymbol{t}) \qquad (1 \le a \le n,\ j > 0). \tag{2.9}$$

In what follows, we shall assume the existence of the Gauss decomposition with respect to the homogeneous gradation:

$$g(z;\boldsymbol{t}) = \{g_{<0}(z;\boldsymbol{t})\}^{-1} g_{\ge 0}(z;\boldsymbol{t}), \tag{2.10}$$

$$g_{<0}(z;\boldsymbol{t}) = 1 + g_{-1} + g_{-2} + \cdots \in \widehat{G}_{<0}, \tag{2.11}$$

$$g_{\ge 0}(z;\boldsymbol{t}) = g_0(1 + g_1 + g_2 + \cdots) \in \widehat{G}_{\ge 0}. \tag{2.12}$$

A detailed discussion about this assumption is in [BtK1, W] for instance.

Lemma 1. *Suppose that $g \in \widehat{G}$ is invertible and the Gauss decomposition* (2.10) *is achieved. Then the decomposition is unique.*

A proof can be found in [KK1], which is an analogue of the theorem 4.1 in [UT] that treats the category of infinite matrices.

From (2.9) and (2.10), we have the following equations for $g_{<0} = g_{<0}(z;\boldsymbol{t})$ and $g_{\ge 0} = g_{\ge 0}(z;\boldsymbol{t})$:

$$\frac{\partial g_{<0}}{\partial t_j^a} = -\left(g_{<0}\Lambda_j^a g_{<0}^{-1}\right)_{<0} g_{<0} = B_j^a g_{<0} - g_{<0}\Lambda_j^a, \tag{2.13}$$

$$\frac{\partial g_{\ge 0}}{\partial t_j^a} = B_j^a g_{\ge 0} \quad (1 \le a \le n,\ j > 0), \tag{2.14}$$

where $B_j^a = B_j^a(z;t)$ is defined by

$$B_j^a \stackrel{\text{def}}{=} \left(g_{<0}\Lambda_j^a g_{<0}^{-1}\right)_{\ge 0} \in \widehat{\mathfrak{g}}_{\ge 0}. \tag{2.15}$$

We call (2.13) and (2.14) the Sato-Wilson equations. Note that the equation (2.13) imply the condition:

$$\partial g_{<0} \stackrel{\text{def}}{=} \sum_{a=1}^{n} \frac{\partial g_{<0}}{\partial t_1^a} = 0, \tag{2.16}$$

and hence the hierarchy of equations (2.13) is equivalent to the $(1,\ldots,1)$-reduction of the n-component KP hierarchy [UT].

The compatibility conditions for (2.13) or (2.14) give rise to the Zakharov-Shabat (or zero-curvature) equations,

$$\left[\frac{\partial}{\partial t_j^a} - B_j^a,\ \frac{\partial}{\partial t_k^b} - B_k^b\right] = 0 \qquad (1 \le a, b \le n, \quad j, k > 0) \tag{2.17}$$

The family of equations of (2.17) gives a hierarchy of soliton equations.

We define the Baker-Akhiezer functions with complex parameters $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\vec{\beta} = (\beta_1, \dots, \beta_n)$ by

$$\Psi^{(\infty)}(z; \boldsymbol{t}, \vec{\alpha}) \stackrel{\text{def}}{=} g_{<0}(z; \boldsymbol{t}) \cdot \Psi_0(z; \boldsymbol{t}) \cdot z^{D(\vec{\alpha})}, \tag{2.18}$$

$$\Psi^{(0)}(z; \boldsymbol{t}, \vec{\beta}) \stackrel{\text{def}}{=} g_{\geq 0}(z; \boldsymbol{t}) \cdot z^{D(\vec{\beta})}. \tag{2.19}$$

Here we have set $D(\vec{\alpha}) = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$ and $D(\vec{\beta}) = \operatorname{diag}(\beta_1, \dots, \beta_n)$. Both of the functions $\Psi^{(\infty)}$ and $\Psi^{(0)}$ satisfy the following equations

$$\partial \Psi \stackrel{\text{def}}{=} \sum_{a=1}^{n} \frac{\partial \Psi}{\partial t_1^a} = z\Psi, \qquad \frac{\partial \Psi}{\partial t_j^a} = B_j^a \Psi \qquad (1 \leq a \leq n, \ j > 0) \tag{2.20}$$

by definitions and the Sato-Wilson equations (2.13) and (2.14).

To obtain the three-wave resonant system (1.1), we now restrict ourselves to the $\widehat{\mathfrak{gl}}_3$ case. We introduce the following parameterizations for the matrices g_{-1} of (2.11) and g_0 of (2.12):

$$g_{-1} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} z^{-1}, \qquad g_0 = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{2.21}$$

Using these parameterizations, we can express B_1^a $(a = 1, 2, 3)$ as follows:

$$B_1^1 = \begin{pmatrix} z & -w_{12} & -w_{13} \\ w_{21} & 0 & 0 \\ w_{31} & 0 & 0 \end{pmatrix}, \tag{2.22}$$

$$B_1^2 = \begin{pmatrix} 0 & w_{12} & 0 \\ -w_{21} & z & -w_{23} \\ 0 & w_{32} & 0 \end{pmatrix}, \tag{2.23}$$

$$B_1^3 = \begin{pmatrix} 0 & 0 & w_{13} \\ 0 & 0 & w_{23} \\ -w_{31} & -w_{32} & z \end{pmatrix}. \tag{2.24}$$

The Zakharov-Shabat equation (2.17) gives the following equations for w_{ij} $(i \neq j)$:

$$\frac{\partial w_{23}}{\partial t_1^1} = w_{21} w_{13}, \qquad \frac{\partial w_{32}}{\partial t_1^1} = w_{31} w_{12}, \tag{2.25}$$

$$\frac{\partial w_{13}}{\partial t_1^2} = w_{12} w_{23}, \qquad \frac{\partial w_{31}}{\partial t_1^2} = w_{32} w_{21}, \tag{2.26}$$

$$\frac{\partial w_{12}}{\partial t_1^3} = w_{13} w_{32}, \qquad \frac{\partial w_{21}}{\partial t_1^3} = w_{23} w_{31}. \tag{2.27}$$

Moreover, w_{ij} satisfy

$$\partial w_{ij} = 0, \qquad \partial = \frac{\partial}{\partial t_1^1} + \frac{\partial}{\partial t_1^2} + \frac{\partial}{\partial t_1^3}, \tag{2.28}$$

because of (2.16). We remark that this condition corresponds to the $(1,1,1)$-reduction of the 3-component KP hierarchy and has been discussed in [KvdL]. We restrict the time variables to the plane,

$$t_1^k = a_k x + b_k \tau \qquad (k=1,2,3), \tag{2.29}$$

where a_k, b_k are complex parameters. Under these conditions, we arrive at the three-wave resonant system (1.1) by setting $u_1 = w_{23}$, $u_1^* = w_{32}$, $u_2 = w_{31}$, $u_2^* = w_{13}$, $u_3 = w_{12}$, $u_3^* = w_{21}$, and applying suitable scaling.

3 Similarity reduction to Schlesinger system

3.1 Extended scaling symmetry and similarity reduction

In this subsection, we consider scaling symmetry of the $\widehat{\mathfrak{gl}}_n$ homogeneous hierarchy. For $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\vec{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}^\times$, we define $\tilde{g}_{<0}(z;\boldsymbol{t})$, $\tilde{g}_{\geq 0}(z;\boldsymbol{t})$ and $\boldsymbol{t}_\lambda$ as follows:

$$\tilde{g}_{<0}(z;\boldsymbol{t}) \overset{\text{def}}{=} \lambda^{D(\vec{\alpha})} g_{<0}(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{-D(\vec{\alpha})}, \tag{3.1}$$

$$\tilde{g}_{\geq 0}(z;\boldsymbol{t}) \overset{\text{def}}{=} \lambda^{D(\vec{\alpha})} g_{\geq 0}(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{-D(\vec{\beta})}, \tag{3.2}$$

$$\boldsymbol{t}_\lambda \overset{\text{def}}{=} (\lambda^j t_j^1, \ldots, \lambda^j t_j^n)_{j>0}. \tag{3.3}$$

We remark that one can set $\beta_n = -1$ without loss of generality. This choice of β_n is important to obtain the Schlesinger system.

Proposition 1. *If $g_{<0}$ and $g_{\geq 0}$ solve the Sato-Wilson equations* (2.13) *and* (2.14)*, then $\tilde{g}_{<0}(z;\boldsymbol{t})$ and $\tilde{g}_{\geq 0}(z;\boldsymbol{t})$ also satisfy* (2.13) *and* (2.14)*.*

Proof. Because of the relation $\Psi_0(\lambda^{-1}z;\boldsymbol{t}_\lambda) = \Psi_0(z;\boldsymbol{t})$ and $[\lambda^{D(\vec{\alpha})}, \Lambda_j^a] = 0$, we find that

$$\lambda^{D(\vec{\alpha})} g(\lambda^{-1}z, \boldsymbol{t}_\lambda)\lambda^{-D(\vec{\beta})} = \Psi_0(z;\boldsymbol{t})\lambda^{D(\vec{\alpha})} g(\lambda^{-1}z, 0)\lambda^{-D(\vec{\beta})}. \tag{3.4}$$

Therefore the function (3.4) satisfies the differential equation (2.9). Due to the uniqueness of the the Gauss decomposition, we have the desired result. □

Proposition 1 is an extension of the Virasoro symmetry of the generalized Drinfel'd-Sokolov hierarchy [BdGHM]. In the case of $n=3$, the parameters $\vec{\alpha}$ and $\vec{\beta}$ correspond to those of the Painlevé VI (See (3.42) below).

We now impose a constraint to the initial data $g(z;0)$ as

$$g(z;0) = \lambda^{D(\vec{\alpha})} g(\lambda^{-1}z;0)\lambda^{-D(\vec{\beta})}. \tag{3.5}$$

This constrained initial data gives the following self-similar solutions of the equations (2.9), (2.13) and (2.14):

$$g(z;\boldsymbol{t}) = \lambda^{D(\vec{\alpha})} g(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{-D(\vec{\beta})} \tag{3.6}$$

$$g_{<0}(z;\boldsymbol{t}) = \lambda^{D(\vec{\alpha})} g_{<0}(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{-D(\vec{\alpha})}, \tag{3.7}$$

$$g_{\geq 0}(z;\boldsymbol{t}) = \lambda^{D(\vec{\alpha})} g_{\geq 0}(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{-D(\vec{\beta})}. \tag{3.8}$$

Proposition 2. *Under the similarity condition* (3.5)*, the Baker-Akhiezer functions* $\Psi^{(\infty)}(z;\boldsymbol{t},\vec{\alpha})$ *and* $\Psi^{(0)}(z;\boldsymbol{t},\vec{\beta})$ *satisfy the linear differential equations,*

$$z\frac{\partial\Psi}{\partial z}=\Big(D(\vec{\alpha})+\sum_{a=1}^{n}\sum_{j>0}jt_j^aB_j^a\Big)\Psi, \tag{3.9}$$

$$\frac{\partial\Psi}{\partial t_j^a}=B_j^a\Psi\quad(1\le a\le n,\ j>0) \tag{3.10}$$

Proof. Differentiating (3.7) with respect to λ at $\lambda=1$, we obtain the equations

$$z\frac{\partial g_{<0}}{\partial z}=[D(\vec{\alpha}),g_{<0}]+\sum_{a=1}^{n}\sum_{j>0}jt_j^a\frac{\partial g_{<0}}{\partial t_j^a}, \tag{3.11}$$

$$z\frac{\partial g_{\ge0}}{\partial z}=D(\vec{\alpha})g_{\ge0}-g_{\ge0}D(\vec{\beta})+\sum_{a=1}^{n}\sum_{j>0}jt_j^a\frac{\partial g_{\ge0}}{\partial t_j^a}. \tag{3.12}$$

From the definition of the Baker-Akhiezer functions $\Psi^{(\infty)}$ and $\Psi^{(0)}$, we have (3.9), (3.10). □

Note especially that we can obtain the following relation from the grade 0 part of (3.12):

$$g_0^{-1}\Big(D(\vec{\alpha})+\sum_{a=1}^{n}\sum_{j>0}jt_j^aB_{j,0}^a\Big)g_0=D(\vec{\beta}). \tag{3.13}$$

Here $B_{j,0}^a$ denotes the grade 0 part of B_j^a, i.e., $B_{j,0}^a=(g_{<0}\Lambda_j^a g_{<0}^{-1})_0$.

Remark In the case of the Darboux-Egoroff equations [AvdL], initial data $g(z)=g(z;0)$ should satisfy the condition $g^{-1}(z)=g^T(-z)$. It follows that $g_{<0}$ and $g_{\ge0}$ satisfy

$$g_{<0}^{-1}(z;\boldsymbol{t})=g_{<0}^T(-z;\boldsymbol{t}),\quad g_{\ge0}^{-1}(z;\boldsymbol{t})=g_{\ge0}^T(-z;\boldsymbol{t}). \tag{3.14}$$

Especially, we have

$$g_0^T=g_0^{-1},\quad g_{-1}=g_{-1}^T. \tag{3.15}$$

In addition, their formulation of the Virasoro condition does not have the parameters that correspond to $\vec{\alpha}$.

3.2 Transformation to the Schlesinger system

In the following, we set

$$t_i\stackrel{\text{def}}{=}t_1^i,\qquad t_j^i=0\quad(1\le i\le n,\ j>1). \tag{3.16}$$

Then the equation (3.9) is rewritten as

$$z\frac{\partial\Psi}{\partial z}=(zT+V)\Psi, \tag{3.17}$$

where T and V are $n \times n$ matrices given by

$$T \stackrel{\text{def}}{=} \operatorname{diag}(t_1, \ldots, t_n), \qquad V \stackrel{\text{def}}{=} D(\vec{\alpha}) + \sum_{a=1}^{n} t_a B^a_{1,0}. \tag{3.18}$$

Notice that the diagonal elements of V are given by α_i because those of $B^i_{1,0} = [g_{-1}, \Lambda^i_1]$ vanish. From the relation (3.13), we can show that V is diagonalized by g_0:

$$g_0^{-1} V g_0 = D(\vec{\beta}). \tag{3.19}$$

The linear equation (3.17) has a simple pole at $z = 0$ and a double pole at $z = \infty$. The solution $\Psi^{(\infty)}$ corresponds to the fundamental matrix at ∞ and the parameters α_i to the monodromy index at ∞. Another solution $\Psi^{(0)}$ corresponds to the fundamental matrix at 0, and β_i to the monodromy index at 0.

To rewrite the linear problem (3.17) as the Schlesinger system (1.3), we apply the Laplace transformation to the solution $\Psi(z)$ of the linear equation (3.17) [H, M]:

$$\Psi(z) \mapsto \Phi(x) = L[\Psi(z)](x) \stackrel{\text{def}}{=} \int_\gamma e^{-zx} \Psi(z) dz. \tag{3.20}$$

Here the contour γ is chosen to satisfy

$$\int_\gamma \frac{\partial f}{\partial z} dz = 0. \tag{3.21}$$

Under this choice, the transformation (3.20) has the properties,

$$\begin{cases} \left(\dfrac{\partial}{\partial x}\right)^k L[\Psi(z)](x) = L[(-z)^k \Psi(z)], \\ \qquad x^k L[\Psi(z)](x) = L\Big[\left(\dfrac{\partial}{\partial z}\right)^k \Psi(z)\Big]. \end{cases} \tag{3.22}$$

By direct calculation, we see that the transformed solution $\Phi(x)$ satisfies the linear equation,

$$(xI - T)\frac{\partial \Phi(x)}{\partial x} = -(V + I)\Phi(x), \tag{3.23}$$

where I is the unit matrix. Notice that we can rewrite (3.23) as

$$\frac{\partial \Phi(x)}{\partial x} = -\sum_{j=1}^{n} \frac{E_{jj}(V + I)}{x - t_j} \Phi(x), \tag{3.24}$$

which is an $n \times n$ Schlesinger system with $(n+1)$ regular singularities at $t_1, \ldots, t_n, \infty$.

Remember the Baker-Akhiezer function $\Psi^{(0)}(z; \boldsymbol{t}, \vec{\alpha})$ of (2.19) has the form,

$$\Psi^{(0)}(z; \boldsymbol{t}, \vec{\alpha}) = g_{\geq 0}(z) z^{D(\vec{\alpha})} = g_0(1 + g_1 + \cdots) z^{D(\vec{\alpha})}. \tag{3.25}$$

The normalized solution at $z=0$ is obtained as $\tilde{\Psi} = g_0^{-1}\Psi(z;\boldsymbol{t})$. We denote as $\tilde{\Phi}(x;\boldsymbol{t})$ the Laplace transformation of $\tilde{\Psi}$:

$$\tilde{\Phi}(x;\boldsymbol{t}) \stackrel{\text{def}}{=} L[\tilde{\Psi}(z;\boldsymbol{t})] = g_0^{-1}\Phi(x;\boldsymbol{t}). \tag{3.26}$$

We have the following corollary of Proposition 2:

Corollary 1. *The function $\tilde{\Phi}(z;\boldsymbol{t})$ of* (3.26) *satisfies the system of equations*

$$\frac{\partial\tilde{\Phi}(z;\boldsymbol{t})}{\partial x} = \sum_{i=1}^{n}\frac{A_i(\boldsymbol{t})}{x-t_i}\tilde{\Phi}(z;\boldsymbol{t}), \tag{3.27}$$

$$\frac{\partial\tilde{\Phi}(z;\boldsymbol{t})}{\partial t_i} = -\frac{A_i(\boldsymbol{t})}{x-t_i}\tilde{\Phi}(z;\boldsymbol{t}) \quad (1 \le i \le n), \tag{3.28}$$

where we have defined $A_i(\boldsymbol{t})$ $(1 \le i \le n)$ as

$$A_i(\boldsymbol{t}) \stackrel{\text{def}}{=} -g_0^{-1}E_{ii}(V+I)g_0 = -g_0^{-1}E_{ii}g_0(D(\vec{\beta})+I). \tag{3.29}$$

Proof. First equation is obtained by (3.24) since g_0 is independent of z. Now we apply the Laplace transformation to the deformation equation (3.10). The linear equation (3.10) for $j=1$ is reduced to

$$\frac{\partial\Psi}{\partial t_i} = B_1^i\Psi = (\Lambda_1^i + [g_{-1},\Lambda_1^i])\Psi. \tag{3.30}$$

The transformed solution $\Phi = L[\Psi]$ satisfies

$$\frac{\partial\Phi}{\partial t_i} = -E_{ii}\frac{\partial\Phi}{\partial x} + [g_{-1},\Lambda_1^i]\Phi = -\frac{E_{ii}(V+I)}{x-t_i}\Phi + [g_{-1},\Lambda_1^i]\Phi. \tag{3.31}$$

To obtain the corresponding linear equation for $\tilde{\Phi} = g_0^{-1}L[\Psi]$, we prepare a differential equation for g_0 with respect to t_i:

$$\frac{\partial g_0}{\partial t_i} = [g_{-1},\Lambda_1^i]g_0, \tag{3.32}$$

which is the grade 0 part of the Sato-Wilson equation (2.14). Combine (3.32) and (3.31), we have the linear equation for $\tilde{\Phi}$:

$$\frac{\partial\tilde{\Phi}}{\partial t_i} = -g_0^{-1}\frac{\partial g_0}{\partial t_i}g_0^{-1} + g_0^{-1}\frac{\partial\Phi}{\partial t_i} = -\frac{g_0^{-1}E_{aa}(V+I)g_0}{x-t_a}\tilde{\Phi}. \tag{3.33}$$

Thus we have (3.28). □

The system of equations (3.27) and (3.28) give the monodromy preserving deformation of the linear equation (3.27) with respect to t_i $(1 \le i \le n)$. Each of A_i satisfies the condition

$$\operatorname{trace} A_i = -\alpha_i - 1, \quad \det A_i = 0 \quad (1 \le i \le n). \tag{3.34}$$

Moreover the relation (3.13) is reduced to

$$\sum_{i=1}^{n} A_i = -g_0^{-1}(V+I)g_0 = -D(\vec{\beta}) - I. \tag{3.35}$$

Under these conditions, the differential equations (3.27) and (3.28) can be regarded as the Schlesinger system normalized at $x=\infty$.

Hereafter in this section, we set $n=3$ and focus on the $\widehat{\mathfrak{gl}}_3$-case. As mentioned in Section 3.1, we can set $\beta_3=-1$ without loss of generality. With this choice, the coefficients $A_i(\boldsymbol{t})$ $(i=1,2,3)$ have the property $(A_i(\boldsymbol{t}))_{j3}=0$ $(i,j=1,2,3)$. Thus the two dimensional subspace $\{\,{}^t(y_1,y_2,0)\,\}$ is invariant under the action of $A_i(\boldsymbol{t})$ $(i=1,2,3)$ with $\beta_3=-1$. Projecting the equations (3.27), (3.28) to the two dimensional subspace, we obtain the 2×2 Schlesinger system of the form,

$$\frac{\partial Y(x;\boldsymbol{t})}{\partial x} = \left\{\frac{\tilde{A}_1(\boldsymbol{t})}{x-t_1} + \frac{\tilde{A}_2(\boldsymbol{t})}{x-t_2} + \frac{\tilde{A}_3(\boldsymbol{t})}{x-t_3}\right\} Y(x;\boldsymbol{t}), \tag{3.36}$$

$$\frac{\partial Y(x;\boldsymbol{t})}{\partial t_i} = -\frac{\tilde{A}_i(\boldsymbol{t})}{x-t_i} Y(x;\boldsymbol{t}) \quad (i=1,2,3), \tag{3.37}$$

where the 2×2 matrices $\tilde{A}_i(\boldsymbol{t})$ $(i=1,2,3)$ are defined as

$$\tilde{A}_i \overset{\text{def}}{=} -\begin{pmatrix} (g_0^{-1})_{1i} \\ (g_0^{-1})_{2i} \end{pmatrix} \begin{pmatrix} (g_0)_{i1}(\beta_1+1) & (g_0)_{i2}(\beta_2+1) \end{pmatrix}, \tag{3.38}$$

and satisfy the following relations:

$$\begin{aligned} &\det \tilde{A}_i = 0, \quad \operatorname{trace} \tilde{A}_i = \operatorname{trace} A_i = -\alpha_i - 1 \quad (i=1,2,3), \\ &-\tilde{A}_1 - \tilde{A}_2 - \tilde{A}_3 = \begin{pmatrix} \beta_1+1 & 0 \\ 0 & \beta_2+1 \end{pmatrix}. \end{aligned} \tag{3.39}$$

Thus we can show that the eigenvalues of $\tilde{A}_i$ are 0 and $-\alpha_i-1$.

We now introduce new variables ξ and t as follows:

$$\xi \overset{\text{def}}{=} \frac{t_1-x}{t_1-t_2}, \qquad t \overset{\text{def}}{=} \frac{t_1-t_3}{t_1-t_2}. \tag{3.40}$$

We finally obtain

$$\frac{\partial Y}{\partial \xi} = \left(\frac{\tilde{A}_1}{\xi} + \frac{\tilde{A}_2}{\xi-1} + \frac{\tilde{A}_3}{\xi-t}\right) Y, \qquad \frac{\partial Y}{\partial t} = -\frac{\tilde{A}_3}{\xi-t} Y, \tag{3.41}$$

which is equivalent to (1.3). So we can obtain a Painlevé VI transcendent from the formula (1.5). The parameters of the Painlevé VI in this case are given by

$$\begin{aligned} \alpha &= \frac{(\beta_1-\beta_2-1)^2}{2}, \quad \beta = -\frac{(\alpha_1+1)^2}{2}, \\ \gamma &= \frac{(\alpha_2+1)^2}{2}, \quad \delta = \frac{1-(\alpha_3+1)^2}{2}. \end{aligned} \tag{3.42}$$

4 Action of the affine Weyl group $W(A_{n-1}^{(1)})$

In this section, we discuss two actions of the the affine Weyl group $W(A_{n-1}^{(1)})$, along the method developed in [KK1, KK2]. Let s_i $(1 \le i \le n-1)$ be the permutation $(i, i+1)$ and S_i be the corresponding permutation matrix,

$$S_i \stackrel{\text{def}}{=} E_{i,i+1} + E_{i+1,i} + \sum_{j \neq i,i+1} E_{jj}. \tag{4.1}$$

We define the left-action s_i^{L} $(i = 1, \dots, n-1)$ by applying S_i to the initial data $g(0)$ from the left:

$$s_i^{\mathrm{L}} g(\boldsymbol{t}) \stackrel{\text{def}}{=} \Psi_0(z; \boldsymbol{t}) S_i g(0) = S_i \Psi_0(z; s_i(\boldsymbol{t})) g(0) = S_i g(s_i(\boldsymbol{t})), \tag{4.2}$$

where the action of s_i to the time-variables $\boldsymbol{t}$ is defined as

$$s_i(\boldsymbol{t}) \stackrel{\text{def}}{=} (t_j^{s_i(1)}, \dots, t_j^{s_i(n)})_{j>0}. \tag{4.3}$$

Here we have used the relation $S_i \Lambda_j^a S_i^{-1} = \Lambda_j^{s_i(a)}$ $(1 \le a \le n, 1 \le i \le n-1, j > 0)$.

Applying the homogeneous Gauss decomposition (2.10) to $\Psi_0(z; t) S_i g(0)$, we define $s_i^{\mathrm{L}} g_{<0}(\boldsymbol{t})$ and $s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t})$:

$$s_i^{\mathrm{L}} g(\boldsymbol{t}) = \left\{ s_i^{\mathrm{L}} g_{<0}(\boldsymbol{t}) \right\}^{-1} s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t}). \tag{4.4}$$

This decomposition induces an action of s_j on the matrix elements of $g_{\ge 0}$ and $g_{<0}$.

Lemma 2. *Assume that the Gauss decomposition* (4.4) *exists. Then the action of* s_i $(i = 1, \dots, n-1)$ *on* $g_{\ge 0}$ *are represented as the permutation of the* i*-th and the* $(i+1)$*-th raws:*

$$(s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t}))_{jk} = g_{\ge 0}(s_i(\boldsymbol{t}))_{s_i(j)k}. \tag{4.5}$$

Proof. The relation (4.4) can be written as follows:

$$\left\{ s_i^{\mathrm{L}} g_{<0}(\boldsymbol{t}) \right\}^{-1} s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t}) = S_i g(s_i(\boldsymbol{t})) = S_i \left\{ g_{<0}(s_i(\boldsymbol{t})) \right\}^{-1} g_{\ge 0}(s_i(\boldsymbol{t})). \tag{4.6}$$

By multiplying $g_{<0}(s_i(\boldsymbol{t}))$ from the left and $g_{\ge 0}^{-1}(s_i(\boldsymbol{t}))$ from the right, we have

$$\begin{aligned} & g_{<0}(s_i(\boldsymbol{t})) S_i \{ g_{<0}(s_i(\boldsymbol{t})) \}^{-1} \\ & \quad = \underbrace{g_{<0}(s_i(\boldsymbol{t})) \{ s_i^{\mathrm{L}} g_{<0}(\boldsymbol{t}) \}^{-1}}_{\in \mathfrak{g}_{<0}} \cdot \underbrace{s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t}) \{ g_{\ge 0}(s_i(\boldsymbol{t})) \}^{-1}}_{\in \mathfrak{g}_{\ge 0}}. \end{aligned} \tag{4.7}$$

Due to the uniqueness of the Gauss decomposition (4.4), we obtain the left-action for $g_{<0}(\boldsymbol{t})$ and $g_{\ge 0}(\boldsymbol{t})$:

$$s_i^{\mathrm{L}} g_{<0}(\boldsymbol{t}) = \left(g_{<0}(s_i(\boldsymbol{t})) S_i \{ g_{<0}(s_i(\boldsymbol{t})) \}^{-1} \right)_{<0} g_{<0}(s_i(\boldsymbol{t})), \tag{4.8}$$

$$s_i^{\mathrm{L}} g_{\ge 0}(\boldsymbol{t}) = \left(g_{<0}(s_i(\boldsymbol{t})) S_i \{ g_{<0}(s_i(\boldsymbol{t})) \}^{-1} \right)_{\ge 0} g_{\ge 0}(s_i(\boldsymbol{t})) \tag{4.9}$$

Notice that $S_i \in \mathfrak{g}_0 \subset \mathfrak{g}_{\geq 0}$ holds because we consider the homogeneous Gauss decomposition. So the relation $(g_{<0}(s_i(\boldsymbol{t}))S_i\{g_{<0}(s_i(\boldsymbol{t}))\}^{-1})_{\geq 0} = S_i$ holds and thus we have

$$s_a^{\mathrm{L}} g_{\geq 0}(\boldsymbol{t}) = S_a g_{\geq 0}(s_a(\boldsymbol{t})). \tag{4.10}$$

This leads to the desired results. □

Next we consider the symmetry of the similarity solution of the $\widehat{\mathfrak{gl}}_n$ hierarchy. Under the similarity condition (3.6), we have

$$\begin{aligned} s_i^{\mathrm{L}} g(z;\boldsymbol{t}) &= S_i \lambda^{D(\vec{\alpha})} g(\lambda^{-1}z; s_i(\boldsymbol{t}_\lambda))\lambda^{D(\vec{\beta})} \\ &= \lambda^{D(s_i(\vec{\alpha}))} s_i^{\mathrm{L}} g(\lambda^{-1}z;\boldsymbol{t}_\lambda)\lambda^{D(\vec{\beta})}, \end{aligned} \tag{4.11}$$

where the action of s_i to the parameters $\vec{\alpha}$ is defined as

$$s_i(\vec{\alpha}) = (\alpha_{s_i(1)}, \ldots, \alpha_{s_i(n)}). \tag{4.12}$$

Then we see that the left-action of the Weyl group is realized as the permutation of the i-th and the $(i+1)$-th elements of the parameter $\vec{\alpha}$ in the similarity condition (3.6).

The left-action of s_i $(1 \leq i \leq n-1)$ induces a permutation of the coefficients in the Schlesinger system (3.27), (3.28).

Proposition 3. *The left-action of the Weyl group s_i^{L} $(1 \leq i \leq n-1)$ induces a permutation of the coefficient matrices of the equation* (3.27) *as follows:*

$$s_i^{\mathrm{L}}\left(\sum_{j=1}^{n} \frac{A_j}{x-t_j}\right) = \sum_{j=1}^{n} \frac{A_{s_i(j)}}{x-t_{s_i(j)}}. \tag{4.13}$$

Proof. By the definition of A_j (3.29), the (k,l)-element of A_j is written as

$$(A_j)_{kl} = -(g_0^{-1})_{kj} g_{jl}(\beta_l+1) = -\frac{\Delta_{jk}(g_0)}{\det g_0} g_{jl}(\beta_l+1), \tag{4.14}$$

where $\Delta_{jk}(g_0)$ is the (j,k)-cofactor of g_0. By virtue of Lemma 2, we can compute the action of s_i^{L} to $\Delta_{jk}(g_0)$, $\det g_0$ and g_{jl} explicitly:

$$s_i^{\mathrm{L}}(\Delta_{jk}(g_0)) = -\Delta_{s_i(j),k}(g_0), \quad s_i^{\mathrm{L}}(\det g_0) = -\det g_0, \quad s_i^{\mathrm{L}}(g_{jl}) = g_{s_i(j)l}. \tag{4.15}$$

We thus obtain

$$s_i^{\mathrm{L}}(A_j) = A_{s_i(j)}, \tag{4.16}$$

which completes the proof. □

If we focus on the $\widehat{\mathfrak{gl}}_3$-case, the left action of Weyl group in Proposition 3 gives transformations of the solutions of the Painlevé VI equation, which are given by

$$s_1^{\mathrm{L}} : t \mapsto 1-t, \quad y \mapsto 1-y \tag{4.17}$$

$$s_1^{\mathrm{L}} s_2^{\mathrm{L}} s_1^{\mathrm{L}} : t \mapsto \frac{t}{t-1}, \quad y \mapsto \frac{t-y}{t-1} \tag{4.18}$$

These transformations of the Painlevé VI coincide with those included in [O2].

We then consider the action of $s_0 \in W(A_{n-1}^{(1)})$. If we define a matrix S_0 as

$$S_0 \stackrel{\mathrm{def}}{=} zE_{n1} + z^{-1}E_{1n} + \sum_{j=2}^{n-1} E_{jj}, \tag{4.19}$$

the left-action s_0^{L} can be constructed in the same manner. Furthermore, we consider the translations in $W(A_{n-1}^{(1)})$, i.e.,

$$\begin{aligned} T_i &= S_{i+1}S_0S_{i+1}S_i \\ &= zE_{ii} + z^{-1}E_{i+1,i+1} + \sum_{j\neq i,i+1} E_{jj}, \end{aligned} \tag{4.20}$$

for $i = 1, \ldots, n-1$, where we define $S_n = S_1$. The action of T_i $(i = 1, \ldots, n-1)$ to the similarity solution makes a change in the parameters $\vec{\alpha}$, which is equivalent to the Schlesinger transformation.

We can also define the right-action of $s_i \in A_{n-1}$ $(i = 1, \ldots, n-1)$ by using the action of S_i to the initial data $g(0)$ [KK1]. This action causes the permutation of the i-th and the $(i+1)$-th columns of $g_{\geq 0}(\boldsymbol{t})$:

$$(s_i^{\mathrm{R}} g_{\geq 0}(\boldsymbol{t}))_{jk} = g_{\geq 0}(s_i(\boldsymbol{t}))_{js_i(k)}. \tag{4.21}$$

Furthermore, the right-action of s_i $(i = 1, \ldots, n-1)$ to the similarity solution causes the permutation of the i-th and the $(i+1)$-th elements of the parameters $\vec{\beta} = (\beta_1, \ldots, \beta_n)$.

5 Concluding remarks

In this paper, we have established the method for obtaining the Painlevé VI from the $\widehat{\mathfrak{gl}}_3$ hierarchy. In the $\widehat{\mathfrak{gl}}_n$-case, we can obtain the 2×2 Schlesinger system that is associated with the Garnier system [O1] from (3.27), (3.28) by setting $\beta_3 = \beta_4 = \cdots = \beta_n = -1$. Hence our method is valid also for the Garnier system.

We also consider the two actions of the affine Weyl group $W(A_{n-1}^{(1)})$ to the $\widehat{\mathfrak{gl}}_n$ hierarchy along the method used in [KK1, KK2]. The left-action is realized as the permutations of the parameters $\vec{\beta}$. In the case of $n = 3$, these actions corresponds to symmetry properties of the Painlevé VI equation. However, the symmetry discussed in this paper is a subset of the full-symmetry of the Painlevé VI, which is isomorphic to $W(F_4^{(1)})$ [O2]. It may be worthwhile to investigate the meaning of other elements in $W(F_4^{(1)})$, in the context of the $\widehat{\mathfrak{gl}}_3$ hierarchy.

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